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MICROCONVECTION IN WEAK FORCE FIELDS. A NUMERICAL COMPARISON OF TWO MODELS

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Mathematical models describing the thermal gravitational convection in weak force fields are presented. They are the classical Oberbeck–Boussinesq model and a new model by V. V. Pukhnachev for microconvection of an isothermally incompressible fluid. Convective fluid flows in annular domains are investigated in various temperature regimes at the boundaries of and in a varying gravity field. Results of a numerical analysis performed by both models are presented. The qualitative and quantitative differences in the flow characteristics under the action of microaccelerations attainable at a space station are verified.

1. Introduction. The Oberbeck–Boussinesq system of equations is a classical mathematical model for describing the thermal gravitational convection. Based on the analysis of the assumptions made in derivation of this system from the exact equations of continuum mechanics, V. V. Pukhnachev [1] proposed a new model to investigate the convection in domains of small extension, in weak gravity or fast-varying temperature fields. The velocity field in this model is nonsolenoidal. However, in the case of a linear dependence of the specific fluid volume $v = 1/\rho$ on the temperature T, the system obtained is transformed to a system in which the modified velocity vector becomes solenoidal, thus making it possible to introduce a stream function for plane and axisymmetric problems. In a stream function–vorticity formulation, calculations are performed of convective flows in a microacceleration field which vary in magnitude and direction and under various boundary temperature regimes. We used herein the method of calculation of convective flows which was developed for problems of free convection in double-connected domains and was based on the linear model of microconvection [2, 3] and also the method of parametric sweep which was developed by A. F. Voyevodin [4] and which is exactly subject to the no-slip conditions at the boundaries of the domains. The problems of convective motions of silicon, glycerin, and some types of glass in the case where the order of magnitude of a new similarity criterion $gl^3/\nu\chi$ (the Pukhnachev number) is smaller than or equal to unity are investigated numerically. Here g is the acceleration of gravity, l is the characteristic size of the domain, ν is the kinematic viscosity, and χ is the thermal diffusivity.

2. Formulation of the Problem. *New Model.* This model is based on the following assumptions [1, 5]:

(1) the fluid density ρ is a function of its temperature T (the fluid is isothermally incompressible);

(2) the fluid potential energy in the field of gravity forces is much smaller than its internal energy;

(3) dissipation of the kinetic energy in the process of motion is negligibly small;

(4) the dynamic viscosity μ, the thermal conductivity k, and the specific heat c are assumed to be constant.

The system of equations for conservation of mass, momentum, and energy is written in the form

$$\rho_t + V\cdot\nabla\rho + \rho\nabla V = 0,\ \rho(V_t + V\cdot\nabla V) = -\nabla P' + \mu\Delta V + \rho g,\ \rho(T_t + V\cdot\nabla T) = \delta\Delta T. \tag{2.1}$$

Here V is the velocity, $P' = P - \zeta\,\mathrm{div}V$, where P is the modified pressure and ζ is the second coefficient of viscosity, $(\ )_t = \partial(\ )/\partial t$, and $\delta = k/c$.

System (2.1) is closed by the equation of state $\rho = R(T)$, where R is a given function of T.

For $\rho = \rho_*(1+\beta T)^{-1}$, the original system (2.1) is transformed into the following form in which the modified velocity vector $W = V - \beta\chi\nabla T$ becomes solenoidal:

$$\mathrm{div}W = 0,$$

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$$W_t + W\cdot\nabla W + \beta\chi(\nabla T\cdot\nabla W - \nabla W\cdot\nabla T) + \beta^2\chi^2(\Delta T\nabla T - \nabla|\nabla T|^2/2)$$
$$= (1+\beta T)(-\nabla q + \nu\Delta W) - \beta T g, \quad T_t + W\cdot\nabla T + \beta\chi|\nabla T|^2 = (1+\beta T)\chi\Delta T. \qquad (2.2)$$

Here $q = P'/\rho_* - g\cdot x - \beta(\nu-\chi)\chi\Delta T$ is the modified pressure, β is the coefficient of thermal expansion, $\nu = \mu/\rho_*$, and $\chi = \delta/\rho_*$, and ρ_* is the characteristic density.

We find the functions W and T, which satisfy the initial

$$W = W_0(x), \quad T = T_0(x), \quad x\in\Omega, \quad t=0 \qquad (2.3)$$

and boundary

$$W = -\beta\chi\nabla T, \qquad \partial T/\partial n = f(x,t), \qquad x\in\Sigma, \qquad t\in[0,t_*] \qquad (2.4)$$

conditions with

$$\int_\Sigma f(x,t)\,d\Sigma = 0$$

(Ω is the flow domain and Σ is its boundary).

The Oberbeck–Boussinesq Model. Derivation of the Oberbeck–Boussinesq model of the thermal gravitational convection is not discussed in detail (see, for example, [5]). The initial boundary-value problem for the Oberbeck–Boussinesq system [5] is to find the velocity V, the temperature T, and the modified pressure P' which satisfy the following system:

$$\begin{gathered} \operatorname{div} V = 0, \qquad V_t + V\cdot\nabla V = -\nabla P' + \nu\Delta V - \beta T g, \\ T_t + V\cdot\nabla T = \chi\Delta T \qquad (P' = P/\rho_* - g\cdot x) \end{gathered} \qquad (2.5)$$

and the following initial

$$V = 0, \qquad T = T_0(x), \qquad x\in\Omega, \qquad t=0 \qquad (2.6)$$

and boundary conditions:

$$V = 0, \qquad \partial T/\partial n = f(x,t), \qquad x\in\Sigma, \qquad t\in[0,t_*]. \qquad (2.7)$$

3. Numerical Simulation. The unsteady convective motion of fluids in an annular domain is studied numerically. Systems (2.2) and (2.5) are considered in the polar coordinates (r,θ). In variables (ψ,ω), the equations are written in the form

$$\omega_t = \tilde\nu\Delta\omega + f_\omega; \qquad (3.1)$$

$$\Delta\psi = -\omega; \qquad (3.2)$$

$$T_t = \tilde\chi\Delta T + f_T, \qquad (3.3)$$

where the functions f_ω and f_T, the coefficients $\tilde\nu$ and $\tilde\chi$, and also the initial (2.3) and (2.6) and boundary (2.4) and (2.7) conditions are specified for both models as follows.

For the new model, we have

$$f_\omega = -\Big(v\frac{\partial\omega}{\partial r} + \frac{u}{r}\frac{\partial\omega}{\partial\theta}\Big) + \beta\Big[\frac{1}{r}\frac{\partial T}{\partial\theta}\frac{\partial q}{\partial r} - \frac{1}{r}\frac{\partial T}{\partial r}\frac{\partial q}{\partial\theta} + \nu\Big(\frac{\partial T}{\partial r}\Big(\Delta u - \frac{u}{r^2}\Big) - \frac{1}{r}\frac{\partial T}{\partial\theta}\Big(\Delta v - \frac{v}{r^2}\Big)\Big)\Big]$$
$$+\beta\Big(g_\theta\frac{\partial T}{\partial r} - \frac{1}{r}g_r\frac{\partial T}{\partial\theta}\Big) - \beta\chi\Big(\omega\Delta T + \frac{\partial T}{\partial r}\frac{\partial\omega}{\partial r} + \frac{1}{r^2}\frac{\partial T}{\partial\theta}\frac{\partial\omega}{\partial\theta}\Big) - \beta^2\chi^2\Big[\frac{1}{r}\Big(-\frac{\partial T}{\partial r}\frac{\partial\Delta T}{\partial\theta} + \frac{\partial T}{\partial\theta}\frac{\partial\Delta T}{\partial r}\Big)\Big],$$

$$f_T = -\Big(v\frac{\partial T}{\partial r} + \frac{u}{r}\frac{\partial T}{\partial\theta}\Big) - \beta\chi|\nabla T|^2$$

$$\Big[W = (v,u) = \Big(\frac{1}{r}\frac{\partial\psi}{\partial\theta}, -\frac{\partial\psi}{\partial r}\Big), \;\; g_r = g_0\cos(\varepsilon t)\sin\theta, \;\; g_\theta = g_0\cos(\varepsilon t)\cos\theta\Big],$$

$$\tilde\nu = (1+\beta T)\nu, \quad \tilde\chi = (1+\beta T)\chi.$$

The initial conditions

$$t = 0: \qquad \omega = 0, \;\; \psi = 0, \;\; T = T_0,$$

and the boundary conditions for $I = 0$

$$r = R_1: \qquad \psi = 0, \;\; \frac{\partial \psi}{\partial r} = \beta\chi R_1^{-1}\frac{\partial T}{\partial \theta}, \;\; \frac{\partial T}{\partial r} = 0,$$

$$r = R_2: \qquad \psi = -\beta\chi R_2 f(t)\sin\theta, \;\; \frac{\partial \psi}{\partial r} = \beta\chi R_2^{-1}\frac{\partial T}{\partial \theta}, \;\; \frac{\partial T}{\partial r} = f(t)\cos\theta;$$

and for $I = 1$

$$r = R_1: \qquad \psi = -\beta\chi R_1 f(t)\sin\theta, \;\; \frac{\partial \psi}{\partial r} = \beta\chi R_1^{-1}\frac{\partial T}{\partial \theta}, \;\; \frac{\partial T}{\partial r} = f(t)\cos\theta,$$

$$r = R_2: \qquad \psi = 0, \;\; \frac{\partial \psi}{\partial r} = \beta\chi R_2^{-1}\frac{\partial T}{\partial \theta}, \;\; \frac{\partial T}{\partial r} = 0.$$

For the Oberbeck–Boussinesq model, we have

$$f_\omega = -\Big(v\frac{\partial \omega}{\partial r} + \frac{u}{r}\frac{\partial \omega}{\partial \theta}\Big) + \beta\Big(g_\theta\frac{\partial T}{\partial r} - \frac{1}{r}g_r\frac{\partial T}{\partial \theta}\Big), \qquad f_T = -\Big(v\frac{\partial T}{\partial r} + \frac{u}{r}\frac{\partial T}{\partial \theta}\Big)$$

$$\Big[V = (v,u) = \Big(\frac{1}{r}\frac{\partial \psi}{\partial \theta}, -\frac{\partial \psi}{\partial r}\Big), \;\; g_r = g_0\cos(\varepsilon t)\sin\theta, \;\; g_\theta = g_0\cos(\varepsilon t)\cos\theta\Big], \;\; \tilde{\nu} = \nu, \quad \tilde{\chi} = \chi.$$

The initial conditions

$$t = 0: \qquad \omega = 0, \quad \psi = 0, \quad T = T_0,$$

and the boundary conditions for $I = 0$

$$r = R_1: \qquad \psi = 0, \;\; \frac{\partial \psi}{\partial r} = 0, \;\; \frac{\partial T}{\partial r} = 0,$$

$$r = R_2: \qquad \psi = 0, \;\; \frac{\partial \psi}{\partial r} = 0, \;\; \frac{\partial T}{\partial r} = f(t)\cos\theta;$$

and for $I = 1$

$$r = R_1: \qquad \psi = 0, \;\; \frac{\partial \psi}{\partial r} = 0, \;\; \frac{\partial T}{\partial r} = f(t)\cos\theta,$$

$$r = R_2: \qquad \psi = 0, \;\; \frac{\partial \psi}{\partial r} = 0, \;\; \frac{\partial T}{\partial r} = 0.$$

The two types of boundary conditions correspond to two types of temperature regimes: we have the heat transfer through the outer boundary of the domain and the insulation of the inner boundary for $I = 0$ and the opposite situation for $I = 1$.

To solve the problem numerically, the method of calculation of convective flows in doubly-connected domains [2] is used. We introduce a difference grid

$$r_n = R_1 + (n-1)h, \quad n = 1, \dots, N+1, \quad h = (R_2 - R_1)/N;$$

$$\theta_m = (m-1)\alpha, \quad m = 1, \dots, M+1, \quad \alpha = 2\pi/M; \qquad t_k = k\tau, \;\; k = 1,\, 2, \dots\, .$$

A longitudinal-transverse finite-difference scheme for Eqs. (3.1) and (3.3) is written in the following general form:

$$\frac{U^{k+1/2} - U^k}{0.5\tau} = \lambda(\Lambda_1 U^k + \Lambda_2 U^{k+1/2}) + F^{k+1/2},$$

$$\frac{U^{k+1} - U^{k+1/2}}{0.5\tau} = \lambda(\Lambda_1 U^{k+1} + \Lambda_2 U^{k+1/2}) + F^{k+1/2}.$$

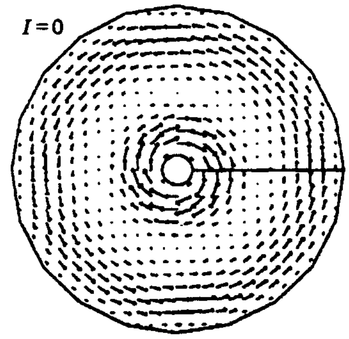

Fig. 1

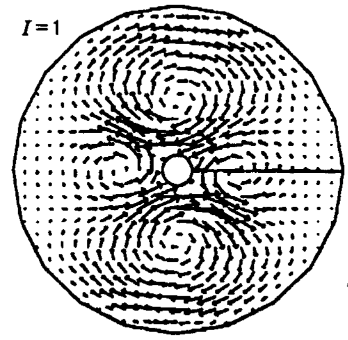

Fig. 2

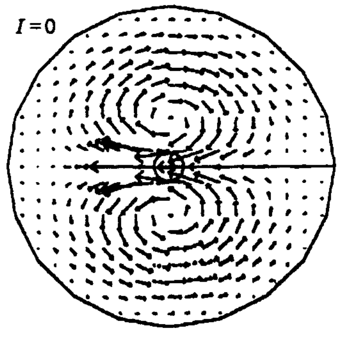

Fig. 3

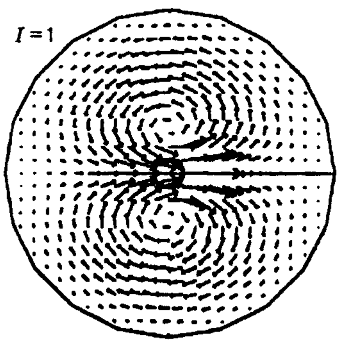

Fig. 4

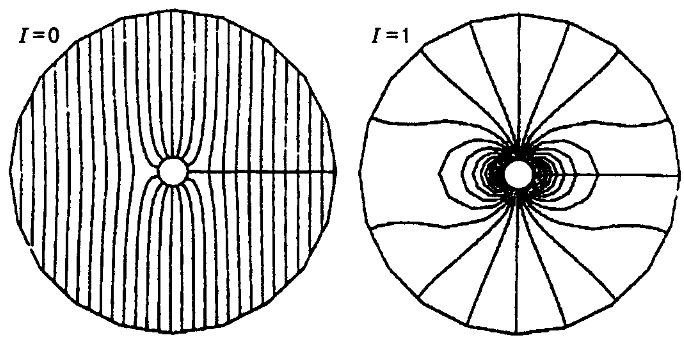

Fig. 5

Here $U^k = U(t^k)$, $U = \binom{\omega}{T}$, and Λ_1 and Λ_2 are the difference operators which approximate, respectively, the differential operators

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}, \quad \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}, \quad \lambda = \tilde{\nu} \quad \text{or} \quad \lambda = \tilde{\chi}.$$

To solve the Poisson equation (3.2) (λ_s is an iteration parameter), the iterative scheme

$$\frac{\psi^{s+1/2} - \psi^s}{0.5\tau} = \lambda_s(\Lambda_1\psi^{s+1/2} + \Lambda_2\psi^s + \omega^{s+1/2}),$$

$$\frac{\psi^{s+1} - \psi^{s+1/2}}{0.5\tau} = \lambda_s(\Lambda_1\psi^{s+1/2} + \Lambda_2\psi^{s+1} + \omega^{s+1/2})$$

is used at each time level.

The method of cyclic sweep is employed to find $T^{k+1/2}$, $\omega^{k+1/2}$, and ψ^{s+1} and the so-called method of parametric sweep [4], according to which $\omega_{n,m} = P_{n,m}\omega_{N+1,m} + Q_{n,m}\omega_{1,m} + R_{n,m}$ and $\psi_{n,m} = \bar{P}_{n,m}\omega_{N+1,m} + \bar{Q}_{n,m}\omega_{1,m} + \bar{R}_{n,m}$, is used to solve the difference equations for ω^{k+1} and $\psi^{s+1/2}$.

4. Results of Numerical Investigation. Calculations of convective flows in a varying field of microaccelerations [$g = g_0\cos(\varepsilon t)$, $g_0 = 10^{-3}$ cm/sec^2, and $\varepsilon = 10^{-1}$ sec^{-1}] are carried out on a 21×21 grid in an annular domain 0.1 cm $= R_1 \leqslant r \leqslant R_2 = 1.1$ cm, $0 \leqslant \theta \leqslant 2\pi$ for silicon, glycerin, and glass.

The values of the Prandtl Pr, Grashof Gr, and Pukhnachev Pu numbers ($\mathrm{Pr} = \nu/\chi$, $\mathrm{Gr} = g\beta T_* l^3/\nu\chi$, and $\mathrm{Pu} = gl^3/\nu\chi$, where T_* is the characteristic temperature difference and $l = R_2 - R_1$) are given in Table 1. The heat transfer through the external or internal boundary of the domain is governed by the following law:

$$\partial T/\partial r = H(t)\cos\theta, \qquad H(t) = (T_1 - T_0)(t/t_1) + T_0, \qquad t \leqslant t_1, \qquad H(t) = T_1, \qquad t > t_1,$$

($t_1 = 60$ sec, $T_0 = 35°$C, $T_1 = 70°$C, and $T_* = T_0$).

The calculations have displayed a qualitative difference in the flow patterns calculated by two different models. This is true primarily for the structure of the flow and its topology and development in time. Figures

TABLE 1

Substance	Pr	Gr	Pu
Silicon	10^{-3}	10^{-4}	1
Glycerin	10^{4}	10^{-3}	10^{-1}
Glass	10^{4}	10^{-6}	10^{-2}

TABLE 2

Substance	Oberbeck–Boussinesq model	New model
	$\lvert u\rvert$, $\lvert v\rvert$, cm/sec	
Silicon	10^{-8}–10^{-6}	10^{-5}–10^{-4}
Glycerin	10^{-9}–10^{-8}	10^{-6}–10^{-5}
Glass	10^{-13}–10^{-10}	10^{-7}–10^{-6}

1–5 show velocity fields and isotherms for silicon at $t = 120$ sec. These patterns are typical of glycerin and glass.

Figures 1–4 demonstrate the velocity field of molten silicon at moment $t = 120$ sec for Pu $= 1$, and Fig. 5 shows the behavior of isotherms under these conditions. Figure 1 (the Oberbeck–Boussinesq model, $I = 0$) shows the velocity field which has the structure of rotational motion with axial symmetry, the external and internal layers of the fluid rotating in different directions. There are two small symmetrically located vortices between these layers: in the upper and lower semicircles for silicon and in the right and left semicircles for glycerin and glass.

The four-vortex flow structure is observed in Fig. 2 (Oberbeck–Boussinesq model, $I = 1$). For silicon, the region occupied by the upper and lower vortices is wider, and, for glycerin and glass, the zones occupied by the right and left vortices are wider.

The two-vortex structure is well discerned in Fig. 3 (new model, $I = 0$). The rotation of the vortex is clockwise in the upper half-plane and anticlockwise in the lower half-plane.

As for $I = 0$, the velocity field is of a two-vortex structure in Fig. 4 (new model, $I = 1$), but the direction of rotation is opposite.

The calculations have shown that there is only a small quantitative difference between the temperature fields obtained by the two different models. Qualitatively, only two types of families of isotherms which correspond to two different types of boundary conditions (Fig. 5, $I = 0$ and $I = 1$) are observed.

The quantitative characteristics, namely, the orders of magnitude of the velocities, are given for all fluids and for two models in Table 2 ($t = 120$ sec).

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